The Monogeneity of Kummer Extensions and Radical Extensions

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Table of contents

Motivation and Background

A Quadratic Field

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$$\frac{279131255861}{371131200000} b_1^{19} + \frac{139394830991}{371131200000} b_1^{18} + \frac{60448487777}{123710400000} b_1^{17} + \frac{280219029161}{371131200000} b_1^{16} +$$
$$\frac{94145035483}{185565600000} b_1^{15} + \frac{44239217807}{371131200000} b_1^{14} + \frac{4438720949}{46391400000} b_1^{13} + \frac{70969469297}{371131200000} b_1^{12} +$$
$$\frac{2509087807}{371131200000} b_1^{11} + \frac{56229143}{2577300000} b_1^{10} + \frac{113716751}{123710400000} b_1^{9} + \frac{22518667}{92782800000} b_1^{8} +$$
$$\frac{3810863}{371131200000} b_1^{7} + \frac{51769603}{371131200000} b_1^{6} + \frac{44967809}{185565600000} b_1^{5} + \frac{1736227}{185565600000} b_1^{4} +$$
$$\frac{3749}{371131200000} b_1^{3} + \frac{1}{966487500} b_1^{2} + \frac{1}{161081250} b_1 + \frac{1}{26846875}, \frac{2722605997}{24742080000} b_1^{19} +$$
$$\frac{7264409407}{24742080000} b_1^{18} + \frac{2635174187}{2749120000} b_1^{17} + \frac{6255406393}{24742080000} b_1^{16} + \frac{168842561}{224928000} b_1^{15} +$$
$$\frac{2269014439}{24742080000} b_1^{14} + \frac{52199291}{386595000} b_1^{13} + \frac{2534812681}{24742080000} b_1^{12} + \frac{910778831}{24742080000} b_1^{11} + \frac{216703}{6248000} b_1^{10} +$$
$$\frac{3915709}{2749120000} b_1^{9} + \frac{423989}{6185520000} b_1^{8} + \frac{1248439}{24742080000} b_1^{7} + \frac{2807459}{24742080000} b_1^{6} + \frac{38131}{224928000} b_1^{5} +$$
$$\frac{729779}{12371040000} b_1^{4} + \frac{13}{24742080000} b_1^{3} + \frac{1}{343640000} b_1^{2} + \frac{1}{128865000} b_1, \frac{119802319}{168696000} b_1^{19} +$$
$$\frac{6689293}{42174000} b_1^{18} + \frac{5183347}{12780000} b_1^{17} + \frac{28338223}{42174000} b_1^{16} + \frac{168250549}{168696000} b_1^{15} + \frac{18297679}{168696000} b_1^{14} +$$
$$\frac{29305517}{168696000} b_1^{13} + \frac{126539399}{843480000} b_1^{12} + \frac{29777}{1917000} b_1^{11} + \frac{28789}{5112000} b_1^{10} + \frac{4073}{11246400} b_1^{9} +$$
$$\frac{9607}{33739200} b_1^{8} + \frac{13711}{76680000} b_1^{7} + \frac{3991}{84348000} b_1^{6} + \frac{57239}{168696000} b_1^{5} + \frac{929}{21087000} b_1^{4} +$$
$$\frac{1}{168696000} b_1^{3} + \frac{1}{281160000} b_1^{2}, \frac{86803537}{95040000} b_1^{19} + \frac{50731939}{95040000} b_1^{18} + \frac{1954979}{2112000} b_1^{17} +$$
$$\frac{14813297}{19008000} b_1^{16} + \frac{5502451}{9504000} b_1^{15} + \frac{18005539}{95040000} b_1^{14} + \frac{919681}{11880000} b_1^{13} + \frac{298123}{1728000} b_1^{12} +$$
$$\frac{26501}{1728000} b_1^{11} + \frac{607}{36000} b_1^{10} + \frac{52387}{31680000} b_1^{9} + \frac{5903}{23760000} b_1^{8} + \frac{6311}{19008000} b_1^{7} + \frac{667}{19008000} b_1^{6} +$$
$$\frac{3401}{9504000} b_1^{5} + \frac{1319}{47520000} b_1^{4} + \frac{1}{95040000} b_1^{3}, \frac{4842}{6875} b_1^{19} + \frac{2683}{13750} b_1^{14} + \frac{7}{13750} b_1^{9} +$$
$$\frac{1}{13750} b_1^{4}, \frac{702}{1375} b_1^{19} + \frac{19}{25} b_1^{18} + \frac{7}{25} b_1^{17} + \frac{1891}{2750} b_1^{15} + \frac{11}{125} b_1^{14} + \frac{1}{25} b_1^{13} + \frac{3}{25} b_1^{12} +$$

A Kummer Extension

Can we write the ring of integers of $\mathbb{Q}\left(\zeta_5, \sqrt[5]{23}\right)$, denoted $\mathcal{O}_{\mathbb{Q}\left(\zeta_5, \sqrt[5]{23}\right)}$, as $\mathbb{Z}[\alpha]$ for some α?

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When are Kummer extensions (and more generally radical, $\sqrt[n]{\bullet}$, extensions) monogenic?

Results

Main Result for Kummer Extensions

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$$\alpha^p \equiv \alpha \bmod (1-\zeta_p)^2 \tag{1}$$

is not *satisfied.*

Main Result for Kummer Extensions

Marie-Nicole Gras[1] has shown that the only monogenic cyclic extensions of $\mathbb{Q}$ of prime degree ≥ 5 are maximal real subfields of cyclotomic fields.

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Over $\mathbb{Q}(\zeta_p)$, however, we can construct infinitely many cyclic extensions of degree p that are monogenic.

Specifically, $\mathbb{Q}\left(\zeta_p, \sqrt[p]{\beta(1-\zeta_p)}\right)$ is monogenic over $\mathbb{Q}(\zeta_p)$ with generator $\sqrt[p]{\beta(1-\zeta_p)}$ for any square-free β that is prime to $1-\zeta_p$.

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Theorem (Smith)

The ring of integers of $L(\sqrt[n]{\alpha})$ is $\mathcal{O}_L[\sqrt[n]{\alpha}]$ if and only if α is square-free as an ideal of $\mathcal{O}_L$ and every prime $\mathfrak{p}$ dividing n **does not** *satisfy Congruence* (2)*.*

Non-monogeneity of Kummer Extensions

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Denote $\mathbb{Q}(\zeta_n, \sqrt[n]{\alpha})$ *by* K*, and suppose there exists a rational prime* ℓ *such that* $\ell \equiv 1 \bmod n$ *and* $\ell < n \cdot \phi(n)$*.*

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Proof Ideas and New Ingredients

Dedekind's Splitting Criterion

Theorem

Let $f(x) \in \mathbb{Z}[x]$ *be monic and irreducible, let* θ *be a root, and let* $L = \mathbb{Q}(\theta)$ *be the number field generated by* θ.

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$$f(x) \equiv \varphi_1(x)^{e_1} \cdots \varphi_r(x)^{e_r} \bmod p$$

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is a factorization of $\overline{f(x)}$ into irreducibles in $\mathbb{F}_p[x]$, then p factors into primes in $\mathcal{O}_L$ as

$$p = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}.$$

Moreover, the residue class degree of $\mathfrak{p}_i$ is equal to the degree of φ_i.

Dedekind's Index Criterion

Theorem (Dedekind[2])

Let $f(x)$ be a monic, irreducible polynomial in $\mathbb{Z}[x]$, θ a root of f, and $L = \mathbb{Q}(\theta)$.

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Let $f(x)$ be a monic, irreducible polynomial in $\mathbb{Z}[x]$, θ a root of f, and $L = \mathbb{Q}(\theta)$. If p is a rational prime, we have

$$f(x) \equiv \prod_{i=1}^{r} f_i(x)^{e_i} \bmod p,$$

where the $f_i(x)$ are monic lifts of the irreducible factors of $\overline{f(x)}$ to $\mathbb{Z}[x]$.

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$$d(x) := \frac{f(x) - \prod_{i=1}^{r} f_i(x)^{e_i}}{p}.$$

Then p divides $[\mathcal{O}_L : \mathbb{Z}[\theta]]$ if and only if $\gcd\left(\overline{f_i(x)}^{e_i - 1}, \overline{d(x)}\right) \neq 1$ for some i, where we are taking the greatest common divisor in $\mathbb{F}_p[x]$.

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Relating Monogeneity and Ramification

Lemma (Smith)

Let L be a number field, $f \in \mathcal{O}_L[x]$ a monic, irreducible polynomial, and θ a root of f.

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Let L be a number field, $f \in \mathcal{O}_L[x]$ a monic, irreducible polynomial, and θ a root of f. Let M be a finite extension of L. Suppose that $f(x)$ is irreducible in $M[x]$ and M is unramified over L at all the primes dividing Δ_f.

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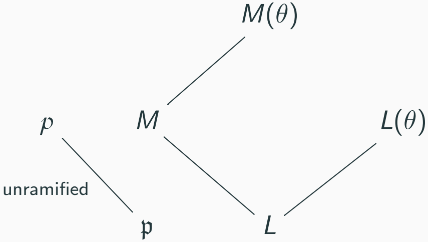
Idea: Extensions that are unramified at the primes dividing Δ_f don't affect the monogeneity of $f(x)$.

The setup of previous theorem is summarized below.



Further Questions

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Are there further insights from a sheaf-theoretic perspective on these results?

Thank You

Thank you for listening. Please send me an email at hanson.smith@colorado.edu if you have any questions that aren't answered here.

A preprint is available on my website, http://math.colorado.edu/~hwsmith/research.html, and on the arXiv at https://arxiv.org/abs/1909.07184.